

# COMPLETE QUANTUM THERMODYNAMICS OF THE BLACK BODY PHOTON GAS

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## Abstract

Kelly and Leff demonstrated and discussed formal and conceptual similarities between basic thermodynamic formulas for the classical ideal gas and black body photon gas. Leff pointed out that thermodynamic formulas for the photon gas cannot be deduced completely by thermodynamic methods since these formulas hold two characteristic parameters,  $r$  and  $b$ , whose accurate values can be obtained exclusively by accurate methods of the quantum statistics (by explicit use of the Planck's or Bose-Einstein distribution). In this work we prove that the complete quantum thermodynamics of the black body photon gas can be done by simple, thermodynamic (non-statistical) methods. We prove that both mentioned parameters and corresponding variables (photons number and pressure) can be obtained very simply and practically exactly (with relative error about few percent), by non-statistical (without any use of the Planck's or Bose-Einstein distribution), quantum thermodynamic methods. Corner-stone of these methods represents a quantum thermodynamic stability condition that is, in some degree, very similar to quantum stability condition in the Bohr quantum atomic theory (de Broglie's interpretation of the Bohr quantization postulate). Finally, we discuss conceptual similarities between black body photon gas entropy and Bekenstein-Hawking black hole entropy.

Kelly [1] and Leff [2] demonstrated and discussed formal and conceptual similarities between basic thermodynamic formulas for the classical ideal gas and black body photon gas. Leff pointed out that thermodynamic formulas for the photon gas cannot be deduced completely by thermodynamic methods since these formulas hold two characteristic parameters,  $r$  and  $b$ , whose accurate values can be obtained exclusively by accurate methods of the quantum statistics (by explicit use of the Planck's or Bose-Einstein distribution). In this work we shall prove that the complete quantum thermodynamics of the black body photon gas can be done

by simple, thermodynamic (non-statistical) methods. Precisely, we shall prove that both mentioned parameters and corresponding variables (photon number and pressure) can be obtained very simply and practically exactly (with relative error about few percent), by non-statistical (without any use of the Planck's or Bose-Einstein distribution), quantum thermodynamic methods. Corner-stone of these methods represents a quantum thermodynamic stability condition which is, in some degree, very similar to quantum stability condition in the Bohr quantum atomic theory (de Broglie's interpretation of the Bohr quantization postulate). Finally, we discuss conceptual similarities between black body photon gas entropy and Bekenstein-Hawking black hole entropy.

As it is well-known [3]-[5] quantum statistics of the black body photon gas predicts accurately, using Planck's or Bose-Einstein distribution, the following expressions for black body photon numbers -  $N$ , internal energy -  $U$ , pressure -  $p$  and entropy -  $S$

$$N = \frac{16\pi k^3 \zeta(3)}{h^3 c^3} T^3 V = r V T^3 \quad (1)$$

$$U = \frac{8\pi^5 k^4}{15 h^3 c^3} T^4 V N = b V T^4 = 2.7 N k T \quad (2)$$

$$p = \frac{1}{3} \frac{U}{V} = \frac{8\pi^5 k^4}{45 h^3 c^3} T^4 = \frac{1}{3} b T^4 = 0.9 \frac{N k T}{V} \quad (3)$$

$$S = \frac{4}{3} \frac{U}{T} = \frac{32\pi^5 k^4}{45 h^3 c^3} T^3 V = \frac{4}{3} b V T^3 = 3.6 N k. \quad (4)$$

Here  $k$  represents the Boltzmann constant,  $h$  - Planck constant,  $c$  - speed of light,  $T$  - black body temperature,  $V$  - black body volume and  $\zeta(3) \simeq 1.202$  - Riemann zeta function, while

$$r = 60.4 \left( \frac{k}{hc} \right)^3 \quad (5)$$

$$b = \frac{8\pi^5 k^4}{15 h^3 c^3} \quad (6)$$

represent parameters that can not be obtained by relatively inaccurate thermodynamic methods but that can be obtained exactly by accurate methods of the quantum statistics (with explicit use of the Planck's or Bose-Einstein distribution).

However, suppose that photon gas is captured in a hollow, spherical black body with relatively large radius  $R$  at temperature  $T$ .

Suppose that, in the thermodynamic equilibrium, photons within hollow, spherical black body propagate, practically, over black body internal surface. Precisely, suppose that these photons propagate over a black body great internal circle with radius  $R$  and circumference  $2\pi R$ , according to the following quantum thermodynamic stability condition

$$\langle m \rangle c R = \langle n \rangle \frac{h}{2\pi}. \quad (7)$$

Here  $\langle n \rangle$  represents the thermodynamically averaged quantum number or (roughly speaking) quantum state and  $\langle m \rangle$  - thermodynamically averaged value of the (single) photon mass in this quantum state.

Expression (7) implies

$$2\pi R = \langle n \rangle \frac{h}{\langle m \rangle c} = \langle n \rangle \langle \lambda \rangle \quad (8)$$

or, correspondingly,

$$R = \langle n \rangle \frac{\langle \lambda \rangle}{2\pi} = \langle n \rangle \langle \lambda_R \rangle . \quad (9)$$

Here  $\langle \lambda \rangle = \frac{h}{\langle m \rangle c}$  represents the thermodynamic average wavelength of the black body photon radiation. It corresponds to the thermodynamically averaged frequency of the black body photon radiation  $\langle \nu \rangle = \frac{c}{\langle \lambda \rangle}$ . Also,  $\langle \lambda_R \rangle = \frac{\langle \lambda \rangle}{2\pi}$  in (9) represents corresponding thermodynamic average reduced wavelength of the black body photon radiation.

In this way quantum thermodynamic stability conditions (7)-(9) simply mean the following. By simple thermodynamic averaging circumference of the black body  $2\pi R$  holds, in the quantum state  $\langle n \rangle$ , corresponding quantum number of the photon wavelengths  $\langle \lambda \rangle$ . Obviously, this condition, in some degree, is similar to quantum stability condition in the Bohr quantum atomic theory (de Broglie's interpretation of the Bohr quantization postulate). But of course, in the Bohr atomic model, that refers on the pure but not on the thermodynamically mixed quantum state, there are different electron circular quantum orbits corresponding to the different quantum states. Vice versa, here different quantum states, dependent of the temperature, correspond to the same photon circular orbit over black body internal surface.

Suppose, according to the Planck formula and (9),

$$\langle \nu \rangle = \frac{kT}{h} \quad (10)$$

so that

$$\langle \lambda \rangle = \frac{hc}{kT} \quad (11)$$

and

$$\langle n \rangle = \frac{R}{\langle \lambda_R \rangle} = \frac{2\pi RkT}{hc}. \quad (12)$$

It can be observed that while  $\langle \nu \rangle$  (10) and  $\langle \lambda \rangle$  (11) represent functions of the  $T$  only,  $\langle n \rangle$  (12) is function of the  $T$  and  $R$ .

Further, expression (9) can be simply transformed in the following expression

$$\frac{4}{3}\pi R^3 = \frac{4}{3} \langle n \rangle^3 \langle \lambda_R \rangle^3 \quad (13)$$

or, since  $\frac{4}{3}\pi R^3$  represents the black body sphere volume  $V$ , in

$$V = \frac{4}{3}\pi \langle n \rangle^3 \langle \lambda_R \rangle^3 . \quad (14)$$

It yields

$$\langle n \rangle^3 = \frac{3}{4\pi} \frac{V}{\langle \lambda_R \rangle^3} \quad (15)$$

or, according to the previously introduced expressions (5), (10), (11) ,

$$\langle n \rangle^3 = 6\pi^2 \left(\frac{k}{hc}\right)^3 T^3 V = 59.2 \left(\frac{k}{hc}\right)^3 V T^3. \quad (16)$$

Now, it can be observed that term  $59.2 \left(\frac{k}{hc}\right)^3$  in (16) is practically (with relative error about 2%) identical to parameter  $r$  (5). It admits statement that

$$r = 6\pi^2 \left(\frac{k}{hc}\right)^3 \quad (17)$$

represents correctly, simply quantum thermodynamically, non-statistically (without any use of the Planck's or Bose-Einstein distribution) derived value of the parameter  $r$ .

Then, right hand of (16) is practically identical to right hand of (5) so that we can suppose that left hand of (16) is practically identical to left hand of (5) too, i.e.

$$N = \langle n \rangle^3. \quad (18)$$

In this way we reproduced correctly, practically accurately (with relative error about 2

Further, it can be simply quantum thermodynamically, non-statistically (without any use of the Planck's or Bose-Einstein distribution), supposed that power,  $P$ , of the black body radiation per unit area,  $A$ , can be approximated by the following expression

$$\frac{P}{A} = \frac{\langle F \rangle c}{A} = \frac{\langle \nu \rangle}{\langle \lambda_R \rangle^2} kT \quad (19)$$

where  $\langle F \rangle$  represents the thermodynamic average force of the back reaction of the radiation at the unit area.

According to (10), (11) and since

$$\frac{\pi^3}{30} = 1.033 \simeq 1 \quad (20)$$

it follows

$$\frac{P}{A} \simeq \frac{\pi^3}{30 \frac{\langle \nu \rangle}{\langle \lambda_R \rangle^2}} kT = \frac{c}{4} b T^4 \equiv \sigma T^4 \quad (21)$$

representing Stefan-Boltzmann law derived simply thermodynamically, non-statistically (without any use of the Planck's or Bose-Einstein statistics), where, of course,  $\sigma$  represents the Stefan-Boltzmann constant. It can be observed that this expression is very close (with relative error about 3%) to the exactly, quantum statistically obtained Stefan-Boltzmann law (by explicit use of the Planck's or Bose-Einstein distribution).

Expression (19) implies the following simple quantum thermodynamic, non-statistical (without any use of the Planck's or Bose-Einstein distribution), expression for the pressure of the photon radiation emitted by black body

$$p_{em} = \frac{\langle F \rangle}{A} = \frac{1}{\langle \lambda \rangle \langle \lambda_R \rangle^2} kT = 4\pi^2 \frac{kT}{\langle \lambda \rangle^3}. \quad (22)$$

It, according to (10), (11), (16), can be simply transformed in the following way

$$p_{em} = 4\pi^2 \frac{kT}{<\lambda>^3} = \frac{4}{6} 6\pi^2 \left(\frac{k}{hc}\right)^3 T^3 kT = \frac{2}{3} \frac{NkT}{V}. \quad (23)$$

On the other hand, according to (9)-(11), simple quantum thermodynamic, non-statistical (without any use of the Planck's or Bose-Einstein distribution), expression for the pressure of the reflected photon radiation in a hollow spherical black body equals approximately

$$p_{ref} = N \frac{\frac{mc^2}{R}}{4\pi R^2} = \frac{1}{3} \frac{NkT}{\frac{4}{3}\pi R^3} = \frac{1}{3} \frac{NkT}{V}. \quad (24)$$

Then, total pressure of the black body photon gas obtained by simple quantum thermodynamic, non-statistical (without any use of the Planck's or Bose-Einstein distribution) methods equals

$$p = p_{em} + p_{ref} = \frac{NkT}{V}. \quad (25)$$

This expression is very close (with relative error about 10

It can be added that when photon gas pressure is determined by simple quantum thermodynamic, non-statistical (without any use of the Planck's or Bose-Einstein distribution) methods (25), parameter  $b$ , according to (3), can be determined by these methods too.

In this way it can be concluded that complete quantum thermodynamics of the black body photon gas can be deduced by simple, quantum thermodynamic, non-statistical (without any use of the Planck's or Bose-Einstein distribution) methods, even if, of course, complete quantum thermodynamics of the photon gas represents an approximate theory of the accurate quantum statistical theory of the photon gas (with explicit use of the Planck's or Bose-Einstein distribution) .

Consider, finally entropy of the black body photon gas according to simple, quantum thermodynamic, non-statistical (without any use of the Planck's or Bose-Einstein distribution) methods. It, in fact, represents introduction of (25) in (4) which, according to (16), yields

$$S = 4kN = k24\pi^2 \left(\frac{kT}{hc}\right)^3 V = k \frac{3}{\pi} \frac{V}{<\lambda_R>^3} \simeq k \frac{V}{<\lambda_R>^3}. \quad (26)$$

Accurate quantum statistical expression for the black body photon gas entropy (4) can be transformed in

$$S = 3.6Nk = \frac{32\pi^5 k^4}{45h^3 c^3} T^3 V = k \left(\frac{4\pi^2}{45}\right) \frac{V}{<\lambda_R>^3} \simeq 0.88k \frac{V}{<\lambda_R>^3}. \quad (27)$$

In both cases, i.e. for (26), (27), entropy of the black body photon gas is practically equivalent to the quotient of the black body volume  $V$  and "minimal quantum thermodynamic volume"  $<\lambda_R>^3$  multiplied by Boltzmann constant  $k$ . It represents very interesting result.

All this is similar to the Bekenstein formula for the black hole entropy

$$S_{BH} = k \frac{A_{BH}}{(2L_P)^2} = k \frac{A_{BH}}{(2\lambda_{RP})^2}. \quad (28)$$

Here  $A_{BH}$  represents the surface of the black hole which, for a spherical, Schwarzschild black hole with horizon radius  $R_S$ , equals  $A_{BH} = 4\pi R_S^2$ . Also, here  $(2L_P)^2$  represents "elementary surface of the black hole" since  $L_P = (\frac{\hbar c}{G})^{\frac{1}{2}} = \lambda_{RP}$  represents the Planck length equivalent to the reduced Compton wave length of a particle with Planck mass, where  $\hbar = \frac{h}{2\pi}$  represents the reduced Planck constant and  $G$  - Newtonian gravitational constant. As it is well-known this Planck length represents the minimal length with the physical meaning in the quantum field theories.

Since entropy represents an additive variable, expression (26) we can transform in the following way

$$S \simeq k \frac{V}{<\lambda_R>^3} \simeq k \frac{V_{shell}}{<\lambda_R>^3} + k \frac{V_{int}}{<\lambda_R>^3} \equiv S_{shell} + S_{int} = k \frac{A}{<\lambda_R>^2} + S_{int}. \quad (29)$$

Here

$$S_{shell} = k \frac{V_{shell}}{<\lambda_R>^3} = k \frac{A <\lambda_R>}{<\lambda_R>^3} = k \frac{A}{<\lambda_R>^2} \quad (30)$$

can be considered as the entropy in a tiny (with width  $<\lambda_R>$ ) spherical shell nearly black body surface  $A = 4\pi R^2$ , while  $S_{int} = S - S_{shell}$  represents the entropy of the rest, internal (under shell) volume of the black body photon gas.

It can be observed that Bekenstein black hole entropy (28) has practically identical form as the entropy of the black body photon gas spherical shell (30). It is not any surprise since black hole represents, at it has been proved by Bekenstein, Hawking and other [6], [7], an especial form of the black body. Classically speaking, according to the cosmic censorship conjuncture, except information on the black hole total mass-energy (for a Schwarzschild black hole), there is no other information on the black hole physical characteristics inside horizon. Quantum field theory, however, roughly speaking, according to the possibility of the particle-antiparticle pair creation nearly horizon, admits additional considerations of the events that occur in  $2\lambda_{RP}$  a tiny spherical shell (with width about  $2\lambda_{RP}$ ) nearly horizon corresponding to the Bekenstein-Hawking black hole entropy as, practically, black body shell entropy. On the other hand, according to the cosmic censorship conjuncture, black hole does not hold internal (under shell) entropy in difference to the other black bodies for which cosmic censorship conjuncture is not satisfied.

In conclusion we can shortly repeat and point out the following. Kelly and Leff demonstrated and discussed formal and conceptual similarities between basic thermodynamic formulas for the classical ideal gas and black body photon gas. Leff pointed out that thermodynamic formulas for the photon gas cannot be deduced completely by thermodynamic methods since these formulas hold two characteristic parameters,  $r$  and  $b$ , whose accurate values can be obtained exclusively by accurate methods of the quantum statistics (by explicit use of the Planck's or Bose-Einstein distribution). In this work we prove that the complete quantum thermodynamics of the black body photon gas can be done by simple, thermodynamic (non-statistical) methods. We prove that both mentioned parameters and corresponding variables (photon number and pressure) can be obtained very simply and practically exactly (with relative error about few percent), by non-statistical (without any use of the Planck's or Bose-Einstein distribution), quantum thermodynamic methods. Corner-stone of these methods represents a quantum thermodynamic stability condition that is, in some degree, very similar to quantum stability

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